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Impatience for Weakly Paretian Orders: Existence and Genericity

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Abstract

We study order theoretic and topological implications for impatience of weakly Paretian, representable orders on infinite utility streams. As a departure from the traditional literature, we do not make any continuity assumptions in proving the existence of impatient points. Impatience is robust in the sense that there are uncountably many impatient points. A general statement about genericity of impatience cannot be made for representable, weakly Paretian orders. This is shown by means of an example. If we assume a stronger sensitivity condition, then genericity obtains.

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1 Introduction

It is widely observed in economic data that in the context of intertemporal decision making almost all economic agents exhibit a preference towards the advancement of timing of future satisfaction. This aspect of human behavior is aptly called impatience. This paper is concerned with the impatience implications of representable, weakly Paretian[1] intertemporal preferences. With regards to impatience, the focus of the literature (Koopmans (1960), Koopmans et.al. (1964)), has been to address two questions:

> *Existence:* For what minimal conditions on a preference order on infinite streams of utility is there some implication of impatience?
>
> *Robustness:* How "many" impatience points are there in the program space?

Time preferences in general, and impatience in particular was discussed by social scientists at least as early as Rae (1834), Bohm-Bawerk (1891) and Fisher (1930). In contrast to the descriptive, albeit compelling discussions along the long history of the topic, a formal analysis on the issue of impatience was first made by Koopmans (1960) and extended by Koopmans et.al. (1964)[2]. Following the important early contributions by Koopmans and his coauthors, subsequent analysis developed precise impatience conditions and obtained clear answers to the question of existence and robustness of impatience. In the words of Brown and Lewis (1981), the basic premise of each study was to

> "....impose as few restrictions as possible .... such that every complete continuous preference relation is (in some precise sense) impatient"[3].

Much of the classical literature on impatience conforms to this line of study. For instance, focusing on the case of continuous (in the sup-metric) preference orders aggregating infinite

[1]The weak Pareto condition states that on ranking infinite utility streams we should prefer a stream $\mathbf{x}$ to $\mathbf{y}$ whenever $\mathbf{x}$ is strictly better than $\mathbf{y}$ in every period. A formal definition is given in section 2.1.

[2]An excellent summary of the classical literature can be found in Koopmans (1972). For a more recent survey of the literature, incorporating impatience as observed in experiments, see Frederick et.al. (2002).

[3]In the interest of full disclosure, the part of the quotation omitted here indicates the exact nature of the topological restriction that Brown and Lewis is after.

utility streams, Diamond (1965) imposed the strong Pareto[4] condition as a fundamental postulate, and showed that if a strongly Paretian preference order is continuous (in the sup metric), then, with an additional non-complementarity axiom, it must exhibit, what he called "eventual impatience". Burness (1973) avoided imposing noncomplementarity axioms in getting his "eventual impatience"result, but he assumed continuously differentiable representation[5].

We view impatience as a strictly behavioral phenomenon and abstain from making any continuity assumption to obtain existence of impatience. This is largely justified by two reasons: (i) continuity in infinite dimensional analysis is sensitive to the choice of topology (unlike the analysis of standard metrics on Euclidean spaces) and (ii) while continuity of a particular evaluation does have behavioral implications; it is, intuitively, a "technical" assumption and as such should be avoided in keeping with the spirit of making "as few restrictions as possible".

To motivate our exploration of minimal sensitivity [6] requirements that guarantee robust impatience, we ask the reverse question. Our first task is to show that there indeed is a strong link between the existence of robust impatience and sensitivity. We start with two very basic requirements. Firstly, we say that any preference order on infinite utility streams must satisfy a basic monotonicity condition (that is, if every generation in one stream is as well off as the same generation in another stream, then we declare the former stream at least as good as the latter) and secondly, that it be representable. In Proposition 1, it is shown that any preference order satisfying these two conditions and additionally, for which the set of impatient points is dense in the sup-metric, must satisfy uniform improvement Pareto (for a formal definition see section 2.3). Having established this connection between existence of robust impatience and sensitivity, we can now ask: *what minimal sensitivity requirement needs to be assumed on a representable order such that the set of impatient points are robust in some precise sense?*

Unfortunately, orders satisfying uniform improvement Pareto do not necessarily exhibit any impatience. This motivates looking beyond uniform improvement Pareto for the minimal sensitivity restrictions for which we can guarantee robust impatience.

[4]The strong Pareto condition states that society should prefer a stream $\mathbf{x}$ to $\mathbf{y}$ whenever period utilities in $\mathbf{x}$ are as good as they are in $\mathbf{y}$, and for some period $\mathbf{x}$ gives a strictly higher utility than $\mathbf{y}$. A formal definition is given in section 2.1.

[5]Burness (1976) studied impatience for separable functions. As will be clear from our motivation and analysis, these conditions are somewhat extraneous to addressing the issue of existence and robustness of impatience.

[6]Sensitivity conditions refer to monotonicity conditions, that is, whether the ranking of streams are sensitive to the ranking of period utilities in binary comparisons. Section 2.1 formalizes all the sensitive conditions used in the paper.

The work of Banerjee and Mitra (2007) provides an intuitive upper bound in our search. We know from their results that every strongly Paretian, representable order must exhibit some impatience and that impatience is generic[7]. Using techniques developed in Dubey and Mitra (2010) we are able to obtain impatience implications based on weaker restrictions than those used in Banerjee and Mitra (2007). We establish a characterization of the domain of infinite utility streams that allow for the existence of some impatience for representable orders satisfying weak Pareto (Theorem 1).

With regards to robustness of impatience we explore two concepts. We can show that the existence result can be strengthened to show that set of impatient points have the power of the continuum (Theorem 2).

Genericity of the set of impatient points will be further evidence of robustness. However for representable, weakly Paretian preference orders the set of impatient points need not be dense; hence, is not generic. This is shown by means of an example. This motivates seeking a sufficient sensitivity condition (stronger than weak Pareto and weaker than strong Pareto) such that impatience is indeed generic (Theorem 3 and Theorem 4). It is worth noting that, like Banerjee and Mitra (2007) our existence result is topology free. We assume continuity of the preference order only in proving that the set of impatient points is open.

The nature of the enquiry in this paper demands that we study impatience implications for a spectrum of sensitivity condition; in the same way as Brown and Lewis (1981) generated impatience implications for continuous orders under different topologies.

The paper is organized as follows. Preliminaries, sensitivity and impatience conditions are introduced in section 2. Existence results are presented in section 4 and the robustness of impatience is established in section 5. We summarize our contributions and relate our findings to existing results after each theorem. This makes a separate concluding section redundant.

[7]Genericity refers to the set of impatient points being open and dense in the sup-metric. In this paper (as in Banerjee and Mitra), denseness does not require continuity of the representable preference but openness does. The genericity result is stated for the case where instantaneous utilities belong to a non-degenerate interval of the real line (for more on the applicability of this case see section 2.1).

2 Preliminaries

2.1 Notation and Order Theoretic Definitions

We will say that a set A is *strictly ordered* by a binary relation R if R is *connected* (if $a, a' \in A$ and $a \neq a'$, then either aRa' or $a'Ra$ holds), *transitive* (if $a, a', a'' \in A$ and aRa' and $a'Ra''$ hold, then aRa'' holds) and *irreflexive* (aRa holds for no $a \in A$). In this case, the strictly ordered set will be denoted by $A(R)$. For example, the set $\mathbb{N}$ is strictly ordered by the binary relation $<$ (where $<$ denotes the usual "less than" relation on the real numbers).

We will say that a strictly ordered set $A'(R')$ is *similar* to the strictly ordered set $A(R)$ if there is a one-to-one function f mapping A onto A', such that:

$$a_1, a_2 \in A \text{ and } a_1 R a_2 \Rightarrow f(a_1) R' f(a_2). \tag{OP}$$

We now specialize to strictly ordered subsets of real numbers. The set of natural numbers will be denoted by $\mathbb{N}$, the set of positive and negative integers by $\mathbb{I}$. We will say that the strictly ordered set $Y(<)$ is of *order type* μ if Y contains a non-empty subset Y' with the property that the strictly ordered set $Y'(<)$ is similar to $\mathbb{I}(<)$[8].

Example 1: Let $a, b \in \mathbb{R}$ with $a < b$. The intervals (denoted by the letter L) (a, b), $[a, b]$, $(a, b]$ and $[a, b)$ are of order type μ. To see this, pick the minimum positive integer N such that $a+(1/N) < b-(1/N)$ and observe that the set $A = \{a+(1/n) : n \geq N \text{ and } n \in \mathbb{N}\} \cup \{b-(1/n) : n \geq N \text{ and } n \in \mathbb{N}\}$ is similar to $\mathbb{I}(<)$ and is contained in L.

As a matter of notation the completion of a proof is denoted by $\blacksquare$, and the completion of a claim is denoted by $\square$. Proofs that are not presented in the body of the paper appear in the appendix.

Instantaneous utilities (also called generational utilities or period utilities) will be assumed to lie in some non-empty subset Y of $\mathbb{R}$. Consequently, infinite utility streams belong to the set X, where $X = Y^{\mathbb{N}}$, the set of all sequences with each term of the sequence being interpreted as one-period utility[9]. If the set of period utilities, Y is specified, the definition of X uniquely

[8] For more details and an excellent exposition of these ideas, see Sierpinski (1965). The terminology of order type μ is from Dubey and Mitra (2010).

[9] Given infinite utility streams $\mathbf{x}, \mathbf{y}$ in X we write $\mathbf{x} >> \mathbf{y}$ if $x_n > y_n$ for all $n \in \mathbb{N}$ and denote by $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

determines the space of infinite utility streams. So we will find it convenient to just describe the set Y, since there should be no confusion about the context of reference.

The set of infinite utility streams $X = Y^{\mathbb{N}}$ with $Y = [0, 1]$ will be of particular interest. We will call this the *classical domain*. It is well known that period utilities in the neoclassical bounded growth model lies in some bounded interval of the real line, see Roy and Kamihigashi (2007) for the one-sector growth model. More generally the reduced form of several dynamic economic models also have the above feature, see Mitra (2000) for a rich set of examples. This case has also been the focus of analysis in the classical papers of Koopmans (1960) and Diamond (1965). Apart from the applicability of this case, it also allows us to compare our existence and robustness results across the spectrum of sensitivity conditions introduced in section 2.3.

An *intertemporal preference order* (interchangeably called a preference order on X) is a binary relation $\succsim$ on X which is complete (if for any $\mathbf{x}, \mathbf{y} \in X$ either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$ holds) and transitive. Given a preference order $\succsim$ on X, we indicate it's asymmetric and symmetric parts by $\succ$ and $\sim$. Recall, for $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \succ \mathbf{y}$ implies $\mathbf{x} \succsim \mathbf{y}$ and *not* $\mathbf{y} \succsim \mathbf{x}$, and the symmetric relation $\mathbf{x} \sim \mathbf{y}$ is defined as $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{x}$.

An intertemporal preference order is *representable* if there is some $U : X \to \mathbb{R}$ such that for any $\mathbf{x}, \mathbf{y} \in X$, we have $\mathbf{x} \succsim \mathbf{y}$ iff $U(\mathbf{x}) \geq U(\mathbf{y})$.

2.2 Topological Preliminaries

For any preference order $\succsim$ on X and any $\mathbf{x} \in X$, denote by $UC(\mathbf{x}) = \{\mathbf{y} \in X : \mathbf{y} \succsim \mathbf{x}\}$ and $LC(\mathbf{x}) = \{\mathbf{z} \in X : \mathbf{x} \succsim \mathbf{z}\}$ the *upper* and *lower contour sets of* $\succsim$ *at* $\mathbf{x}$. An intertemporal preference order $\succsim$ is *continuous* in a topology Υ of X if $UC(\mathbf{x})$ and $LC(\mathbf{x})$ are closed subsets in (X, Υ) for every $\mathbf{x} \in X$.

The analysis of topological results will be with regard to the classical domain, that is for $Y = [0, 1]$. On $X = Y^{\mathbb{N}}$, we define the concept of distance between two points by the sup-metric; that is for $\mathbf{x}, \mathbf{y} \in X$ the metric topology generated by the function $d : X^2 \to \mathbb{R}$ given by $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}$. We will denote the metric space by the tuple (X, d). For any $\epsilon > 0$, denote the open ball around some $\mathbf{x} \in X$ with radius ϵ by $B(\mathbf{x}, \epsilon)$.

Given (X, d), a subset A of X is said to be *generic* if it is dense and open in X.

2.3 Sensitivity Conditions

The fundamental behavioral restriction we impose on intertemporal preferences is that of sensitivity to generational utilities. We present five sensitivity conditions. Let $\succsim$ be a preference order on X, it is said to satisfy

Monotonicity: if $\mathbf{x}, \mathbf{y} \in X$ and $\mathbf{x} \geq \mathbf{y}$, then $\mathbf{x} \succsim \mathbf{y}$,

Uniform Improvement Pareto: if for any $\mathbf{x}, \mathbf{y} \in X$ there is some $\beta > 0$ such that $x_n \geq y_n + \beta$, then $\mathbf{x} \succ \mathbf{y}$,

Weak Pareto: if $\mathbf{x}, \mathbf{y} \in X$ and $\mathbf{x} >> \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$,

Infinite Pareto: if $\mathbf{x}, \mathbf{y} \in X$ and $x_n \geq y_n$ for all $n \in \mathbb{N}$ and for some subsequence $\{N_k\}$ of $\mathbb{N}$ the inequality is strict, then $\mathbf{x} \succ \mathbf{y}$,

Strong Pareto: if $\mathbf{x}, \mathbf{y} \in X$ and $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Note that in each of these conditions an inference is made about the relative ranking of two streams from information on generational utilities. Such an aggregation reflects the sensitivity of the ordering to period utilities; hence the name sensitivity condition. It is easy to verify that the sensitivity conditions, starting with uniform improvement Pareto, become more demanding as we move down the list in the sense that a preference order satisfying Strong Pareto, must satisfy uniform improvement Pareto, weak Pareto and infinite Pareto.

The monotonicity condition is the most basic sensitivity requirement. It requires that we do not reverse the natural weak dominance of vector comparability when we compare two streams. The weak Pareto condition along with the monotonicity condition is S1 in Diamond (1965, p. 172) and the strong Pareto condition is condition S2 in Diamond (1965, p.173). The use of the word Pareto is in reference to the social choice literature; in particular in finite generation social choice the weak Pareto condition has been used extensively, see Arrow (1963) and Sen (1977) for instance. The strong Pareto condition is what is known as strictly increasing, see for example, Benoit and Ok (2007). The infinite Pareto condition is extensively analysed in Crespo, Nuñez and Zapatero (2009); whereas Banerjee (2011) and Sakai (2011) make a compelling case for the uniform improvement Pareto axiom.

We will be exclusively dealing with representable preference orders on X satisfying sensitivity conditions. For representable preference orders there are natural analogues of each of the sensitivity conditions expressed in terms of the function that represents the order.

2.4 Impatience Condition

We provide here a precise definition of what we mean for a preference order on X to exhibit impatience at some $\mathbf{x} \in X$. Some auxiliary definitions are needed to formalize our impatience condition. Given $\mathbf{x} \in X$, and $M, N \in \mathbb{N}$, we denote by $\mathbf{x}\,(M, N)$ the sequence $\mathbf{x}' \in X$ defined by,

$$x'_M = x_N,\ x'_N = x_M \text{ and } x'_n = x_n,\ \forall n \neq N, M. \tag{1}$$

An intertemporal preference order $\succsim$ is said to *exhibits impatience at* $\mathbf{x} \in X$, if there exist $M, N \in \mathbb{N}$ with $M > N$ such that, either

$$\text{(i) } x_M > x_N \text{ and } \mathbf{x}(M, N) \succ \mathbf{x}\text{; or (ii) } x_M < x_N \text{ and } \mathbf{x} \succ \mathbf{x}(M, N). \tag{2}$$

Observe that the definition of a new sequence $\mathbf{x}(M, N)$ in (1) from some $\mathbf{x} \in X$ involves swapping one-period utilities corresponding to periods M and N, ceteris paribus. The impatience condition captures the intuition that the preference order $\succsim$ exhibits a preference towards "immediate gratification".

For representable preference orders on X, the information from the impatience condition (2) can be translated to the real-valued function that represents it. If $\succsim$ is a representable (by a real valued function W) preference order on X and exhibits impatience at $\mathbf{x} \in X$, then there exist $M, N \in \mathbb{N}$ with $M > N$ such that, either

$$\text{(i) } x_M > x_N \text{ and } W(\mathbf{x}(M, N)) > W(\mathbf{x})\text{; or (ii) } x_M < x_N \text{ and } W(\mathbf{x}) \succ W(\mathbf{x}(M, N)).$$

Alternatively, if $W : X \to \mathbb{R}$ represents $\succsim$ and exhibits impatience at $\mathbf{x} \in X$, then there exists $M, N \in \mathbb{N}$ with $M > N$ such that,

$$(x_N - x_M)(W(\mathbf{x}(M, N)) - W(\mathbf{x})) < 0. \tag{3}$$

3 Sensitivity Implications when Impatience is Dense

In this section, we show that there indeed is a strong link between the existence of robust impatience and sensitivity. Precisely, if for some monotone preference order on the classical domain, the set of impatient points pertaining to the order is dense in the sup-metric, then the preference order must satisfy uniform improvement Pareto. Since the monotonicity condition is easy to justify in economic environments, Proposition 1 shows that uniform improvement Pareto is a necessary implication of robust impatience.

Proposition 1 *Suppose $Y = [0,1]$ and $X = Y^{\mathbb{N}}$. Let $\succsim$ be a preference order on X satisfying monotonicity. Assume that the set of points of X at which $\succsim$ exhibits impatience*

$$I = \{\mathbf{x} \in X : \ \succsim \ \textit{exhibits impatience at } \mathbf{x}\}$$

is dense in (X, d). Then $\succsim$ satisfies uniform improvement Pareto.

Proof. Let $\beta > 0$ and $\mathbf{x}, \mathbf{y} \in X$ satisfy $x_n \geq y_n + \beta$ for all $n \in \mathbb{N}$. We have to show that $\mathbf{x} \succ \mathbf{y}$. Since $\succsim$ satisfies monotonicity, $\mathbf{y} \succ \mathbf{x}$ is impossible. Assume by way of contradiction that $\mathbf{x} \sim \mathbf{y}$. Consider the following subset of X

$$A = \{\mathbf{z} \in X : \text{there is } \epsilon \in (0, \beta/2) \text{ such that } z_n + \epsilon \leq y_n + \beta/2 \text{ and } y_n + \epsilon \leq z_n \text{ for all } n \in \mathbb{N}\}.$$

The set A is open in X. To see this let $\mathbf{z} \in A$, then there is $\epsilon \in (0, \beta/2)$ such that $z_n + \epsilon \leq y_n + \beta/2$ and $y_n + \epsilon \leq z_n$ for all $n \in \mathbb{N}$. Observe that $B(\mathbf{z}, \epsilon/2) \subset A$. Since for $\mathbf{w} \in B(\mathbf{z}, \epsilon/2)$, $y_n + \epsilon \leq z_n < w_n + (\epsilon/2) < z_n + \epsilon \leq y_n + \beta/2$ implies $y_n + (\epsilon/2) < w_n$ and $w_n + (\epsilon/2) < y_n + \beta/2$ as is required.

Now for any $\mathbf{z} \in A$ we must have $\mathbf{z} \sim \mathbf{y}$. Since $\mathbf{z} \in A$, the monotonicity condition implies that $\mathbf{y} \succ \mathbf{z}$ is impossible. If however, $\mathbf{z} \succ \mathbf{y}$, then by transitivity of $\succsim$, we have $\mathbf{z} \succ \mathbf{x}$, which is impossible by monotonicity. So we must have $\mathbf{z} \sim \mathbf{y}$ for any $\mathbf{z} \in A$. This implies that for any $\mathbf{z}, \mathbf{z}' \in A$ the relation $\mathbf{z} \sim \mathbf{z}'$ is true, proving that A, an open set in X has no impatient point, contradicting the fact that I is dense in X. Thus, we have shown that $\succsim$ must satisfy uniform improvement Pareto. ■

Remark:

Converse of Proposition 1: Proposition 1 establishes the connection between impatience and sensitivity. Uniform improvement Pareto is the starting point of our analysis of the minimal sensitivity requirement that guarantees the existence of robust impatience. Unfortunately, there is at least one example of a representable, continuous, monotone preference order on the classical domain that exhibits no impatience; the relation $\mathbf{x} \succsim \mathbf{y}$ iff $\sup x_n \geq \sup y_n$ is one such example when $Y = [0,1]$. For more on the possibility of pure patience and the domain restrictions imposed by a purely patient representable preference order satisfying the uniform improvement Pareto condition, see Banerjee (2011).

Representability under the assumptions of Proposition 1: If, in addition to the conditions of Proposition 1, we assumed that $\succsim$ is continuous, then it is easy to show that $\succsim$ must also be representable. Note that a dense I and monotonicity implies uniform improvement Pareto by Proposition 1. Representability of $\succsim$ now follows from noting that the Existence Theorem in Diamond (1965 p. 173) holds under monotonicity and uniform improvement Pareto. We do not know whether representability follows from the assumptions in Proposition 1 alone.

4 Existence and Characterization of Impatient Domains

Proposition 1 and the remark following it indicates: (a) some sensitivity of the preference order is directly implied by the existence of robust impatience, and (b) a sensitivity condition stronger than uniform improvement Pareto is needed to obtain impatience implications in the classical domain. We have been silent about whether representability is necessary in obtaining impatience implication for a sensitive preference order.

In Svensson (1980), it was shown that absent any continuity requirement, one can define preference orders on infinite utility streams that are purely patient. Notably, such purely patient orders are not representable, see Basu and Mitra (2003). This implies that to demonstrate a general result on impatience, representability is indispensable. Theorem 1 shows that strong Pareto can be dispensed with for a weaker sensitivity requirement, weak Pareto, at least when one is concerned with impatience in the classical domain.

This section shows that existence of impatient is a necessary implication for weakly Paretian, representable orders. Theorem 1 goes further in characterizing a precise domain restriction for which weakly Paretian, representable preference orders are indeed impatient.

Our demonstration of the existence of an impatient point is based on the analysis of Dubey and Mitra (2010). We discuss the proof technique informally in the remarks following Theorem 2.

Theorem 1 *Suppose Y is a non-empty subset of $\mathbb{R}$ and let $X = Y^{\mathbb{N}}$. For any representable, weakly Paretian preference order $\succsim$ on X there exists some $\mathbf{x} \in X$ at which $\succsim$ exhibits impatience iff Y is of order type μ.*

Remarks:

Relation to Literature: (i) The impatience condition in our paper is the same as in Banerjee and Mitra (2007). In the classical domain, our result is a generalization of Banerjee and Mitra (2007) by virtue of a weaker sensitivity requirement. In contrast, Diamond (1965) proved the existence of eventual impatience, which captures impatience in utility streams where the tail of the stream is uniformly bounded away from the first period utility. He used the strong Pareto and non-complementarity conditions to obtain his result. Our results are non-comparable to his results on two dimensions; firstly, impatience in our case can obtain in utility streams that are convergent which is not the case in Diamond (1965), secondly, we do not need any non-complementarity, continuity or strong Pareto conditions to obtain our impatience implication. However, it is worth noting that Diamond (1965) achieves a stronger form of impatience implication; he demonstrates that every stream for which first period utilities are bounded away from the tail of the stream must exhibit impatience.

(ii) In a recent paper Benoit and Ok (2007) study "delay aversion". This can be viewed as comparative impatience, in the sense that impatience content of two representable preference orders are compared in their paper. They also address the very interesting and intuitive possibility where a preference towards improving current consumption at the cost of diminished future consumption need not necessarily imply impatience. In a intertemporal setting with endowments, a preference towards the advancement of timing of consumption could be a consequence of a lower endowment today. This issue is not addressed in this paper. It would be interesting to study the questions of this paper in a general equilibrium model incorporating this feature.

5 Robustness of Impatience

5.1 Order Theoretic Implications

We will first establish that the set of impatient points of a representable, weakly Paretian intertemporal preference order has the power of the continuum.

Theorem 2 *Suppose Y is a non-empty subset of $\mathbb{R}$ that is of order type μ and let $X = Y^{\mathbb{N}}$. Let $\succsim$ be a representable, weakly Paretian preference order on X. Then the set of points of X at which $\succsim$ exhibits impatience*

$$I = \{\mathbf{x} \in X : \ \succsim \ \textit{exhibits impatience at } \mathbf{x}\}$$

is uncountable.

Remarks:

Method of Proof: The nature of proofs presented so far bears a resemblance to Dubey and Mitra (2010). They showed the impossibility of anonymous representable preference orders[10] satisfying weak Pareto. The existence of an impatient point for weakly Paretian, representable order is a consequence of their proof technique and is not a direct corollary of their results. We exploit a crucial feature that drive their result; that is, we explore sequences for which positive terms appear along some subsequence in increasing order and along the complementary subsequence negative terms appear in decreasing order. This feature along with the inductive nature of order type μ subsets to which period utilities belong, is sufficient to guarantee the existence of impatience. Our method (again exploiting this feature of sequences) does more than existence; we have shown, using order theoretic methods (without invoking any topological properties) alone, that impatient points are in fact, uncountable.

5.2 Topological Implications

The set of impatience points being of the order of the continuum shows that impatience is robust in a weak sense. In this section we pursue a stronger result.

We show that for every weakly Paretian, representable intertemporal preference order $\succsim$, the set of impatient points is *not* necessarily generic. Precisely, we show (by means of an example) that a representable and weakly Paretian preference order on X exists, for which, the set of impatient points is not dense, and consequently is not generic. However, if we strengthen the sensitivity requirement to infinite Pareto, genericity of the set of impatient points follow.

5.2.1 Impatience is not Generic for Weakly Paretian Orders

In this section we provide an example of a weakly Paretian, representable preference order on X that exhibits no impatience on some open set of (X, d). In particular this implies that the set of impatient points pertaining to this order cannot be a dense, open subset of X.

[10] Anonymity means that ranking of streams is indifferent to the finite permutation of generational utilities as defined in (1).

Example 2: Let $Y = [0,1]$ and $X = Y^{\mathbb{N}}$. Consider the following class of subsets of X:

$$\mathcal{V} = \{V \subset X : V = \textstyle\prod_{i=1}^{\infty} V_i \text{ and there exists some } \textit{minimally} \text{ chosen } N \in \mathbb{N} \text{ s.t. } V_i = [0,1/2) \text{ for all } i > N\}.$$

In the definition of the class $\mathcal{V}$, we choose $N(V)$ to correspond to the *smallest* natural number such that for all integers $i > N(V)$ we have $V_i = [0,1/2)$. It is in this sense that we use the word "minimally chosen" in the definition. It is important to note that for any $V \in \mathcal{V}$, and any $\mathbf{x} \in V$, there is some $t \in \mathbb{N}$ such that $x_t \in [1/2,1]$. Note that the infinite cartesian product of $[0,1/2)$ does *not* belong to this set. Sets in the class $\mathcal{V}$ have the feature that at least some (and at most finitely many) elements are greater than or equal to $(1/2)$.

This class of subsets is non-empty. We will first define a sequence of subsets $\{U_i\}$ of X which is crucial to the demonstration and also establishes non-emptiness of $\mathcal{V}$. Let

$$U_i = [0,1/2) \times [0,1/2) \times \cdots \times [0,1/2) \times \underbrace{(1/2,1]}_{i^{th}\text{place}} \times [0,1/2)^{\mathbb{N}} \tag{4}$$

be the cartesian product of the intervals $[0,1/2)$ and $(1/2,1]$ with the latter interval appearing in the i^{th} position. Observe that $U_i \cap U_j = \emptyset$ for all $i \neq j$. From (4), it is clear that $U_i \in \mathcal{V}$ for each $i \in \mathbb{N}$. It is also easy to see that U_i is an open set in (X,d) for each $i \in \mathbb{N}$. To see this, denote by $\mathbf{x}^i = (0,0,..,0,1,0,..)$, the vector in X with 1 at the i^{th} position and 0 elsewhere and note that $U_i = B(\mathbf{x}^i, 1/2)$. This implies that U_i is open in X. We will write $U = \cup_{i\in\mathbb{N}} U_i$. Clearly, U is open in X.

To facilitate the exposition, for $N \in \mathbb{N}$ denote the set $\{1,2,...,N\}$ by $[N]$.

Define the function $W : X \to \mathbb{R}$ by

$$W(\mathbf{x}) = \begin{cases} \max\{x_n : n \in [N(V)]\} & \text{for } \mathbf{x} \in V \in \mathcal{V} \\ x_1 & \text{for } \mathbf{x} \in [0,1/2)^{\mathbb{N}} \\ \sum_{n=1}^{\infty} (1/2)^{n-1}(2 + x_n) & \text{otherwise.} \end{cases} \tag{5}$$

Now define $\succsim$ as:

$$\mathbf{x} \succsim \mathbf{y} \text{ iff } W(\mathbf{x}) \geq W(\mathbf{y}).$$

Claim 1: $\succsim$ *satisfies weak Pareto.*

Proof: Let $\mathbf{x}' >> \mathbf{x}$ for $\mathbf{x}, \mathbf{x}' \in X$. Consider the following cases: (A) $\mathbf{x}' \in V' \in \mathcal{V}$ (B) $\mathbf{x}' \notin V$ for any $V \in \mathcal{V}$.

In case (A), we must have either (a) $\mathbf{x} \in V \in \mathcal{V}$ or (b) $\mathbf{x} \in [0, 1/2)^{\mathbb{N}}$. In (a), $\mathbf{x}' >> \mathbf{x}$ implies $N(V) \leq N(V')$. Let $j = \arg\max\{x_n : n \in [N(V)]\}$ and $k = \arg\max\{x'_n : n \in [N(V')]\}$. Using the appropriate range in (5), we have $W(\mathbf{x}') = x'_k$ and $W(\mathbf{x}) = x_j$. Note that $\mathbf{x}' >> \mathbf{x}$ and $N(V) \leq N(V')$ implies $W(\mathbf{x}) = x_j < x'_j \leq x'_k = W(\mathbf{x}')$. In (b), $\mathbf{x}' >> \mathbf{x}$ and (5) implies $W(\mathbf{x}) = x_1 < (1/2) \leq W(\mathbf{x}')$.

In case (B), two sub-cases are possible: (i) $\mathbf{x}' \in [0, 1/2)^{\mathbb{N}}$ or (ii) $\mathbf{x}' \notin [0, 1/2)^{\mathbb{N}}$. Observe that in (i), $\mathbf{x}' >> \mathbf{x}$ implies $\mathbf{x} \in [0, 1/2)^{\mathbb{N}}$. Using (5) we get $W(\mathbf{x}) = x_1 < x'_1 = W(\mathbf{x}')$. In sub-case (ii),

$$W(\mathbf{x}) \leq \sum_{n=1}^{\infty} (1/2)^{n-1}(2 + x_n) < W(\mathbf{x}')$$

holds, as was needed. □

Claim 2: *Every point of U (an open set in X) is a patient point of $\succsim$.*

Proof: For any $\mathbf{x} \in U$, there is some (unique) U_i such that $\mathbf{x} \in U_i \in \mathcal{V}$. This implies that $\max\{x_n : n \in \mathbb{N}\} = x_i$. For any $M, N \in \mathbb{N}$ with $M > N$ we must have $\mathbf{x}(M, N) \in U_i$ if $M \neq i$ and $N \neq i$; $\mathbf{x}(M, N) \in U_N$ if $M = i$ or $\mathbf{x}(M, N) \in U_M$ if $N = i$. In each of these cases, $W(\mathbf{x}(M, N)) = W(\mathbf{x})$ holds. □

Thus, we have demonstrated the existence of a real valued function on X that exhibits pure patience on some open set of the program space (X, d). This shows that there is no hope of obtaining a result of genericity of the set of impatient points for weakly Paretian, representable preference orders.

It can also be verified that the preference order $\succsim$ satisfies monotonicity.

We note here (without proof) that W is *not* continuous in the sup-metric. This observation actually leads us to the following open question:

Open Question: Is there a (sup-metric) continuous preference order satisfying weak Pareto and monotonicity with at least one purely patient point?

If the answer to the above open question is a no, then it would have been established that for continuous preference order satisfying weak Pareto and monotonicity, every non-constant stream exhibits impatience. In particular this would make the analysis in section 5.2.2 redundant. We hope that our construction above provides some direction in search of an answer to the open question.

5.2.2 Sufficient Condition for Genericity

In this section, we prove that for representable preference orders satisfying infinite Pareto, impatience is indeed generic. On the one hand, Example 2 demonstrates that genericity is not implied by a weakly Paretian, representable preference order and the analysis in Banerjee and Mitra (2007) guarantees that if we strengthen sensitivity all the way to strong Pareto, then impatience is indeed generic. We are able to establish genericity for representable orders which satisfy a sensitivity requirement between weak and strong Pareto, generalizing the Banerjee-Mitra genericity result.

We first show that the set of impatient points of a representable intertemporal preference order satisfying infinite Pareto must be dense in (X, d).

Theorem 3 *Suppose* $Y = [0, 1]$ *and* $X = Y^{\mathbb{N}}$. *Let* $\succsim$ *be a representable intertemporal preference order satisfying infinite Pareto. Then the set of impatient points of* X *at which* $\succsim$ *exhibits impatience*

$$I = \{x \in X \ : \ \succsim \ \textit{exhibits impatience at } x\}$$

is a dense subset in (X, d).

To show that set I is generic, in addition to Theorem 3 we need to show that I is an open set in (X, d). We will make the additional assumption that $\succsim$ is continuous in (X, d), that is for each $\mathbf{x} \in X$, the upper and lower contour sets $UC(\mathbf{x})$ and $LC(\mathbf{x})$ are both closed in (X, d). The proof of this fact is identical to Theorem 3 in Banerjee and Mitra (2007). In fact, the assumption of continuity of $\succsim$ allows us to drop the representability requirement, since by the Existence Theorem proven in Diamond (1965, p. 173) a continuous preference order on X satisfying infinite Pareto can be represented by a continuous (on (X, d)) function. We state this result without proof.

Theorem 4 *Suppose* $Y = [0, 1]$ *and* $X = Y^{\mathbb{N}}$.*Let* $\succsim$ *be a preference order on* X *satisfying infinite Pareto. Then the set of impatient points of* X *at which* $\succsim$ *exhibits impatience*

$$I = \{x \in X \ : \ \succsim \ \textit{exhibits impatience at } x\}$$

is an open subset in (X, d).

6 Appendix: Existence

This section is dedicated to the proof of Theorem 1. Following the method in Dubey and Mitra (2010), the proof is in two steps. We first establish the existence of an impatient point for weakly Paretian, representable intertemporal orders on $X = Y^{\mathbb{N}}$ with $Y = \mathbb{I}$. The next step extends this result to domains of infinite utility streams where the period utilities belong to non-empty subsets of $\mathbb{R}$ of order type μ. The characterization result, Theorem 1, follows from noting the existence of purely patient preference orders when period utilities belong to a subset of $\mathbb{R}$ that is *not* of order type μ.

To facilitate the proof of the robustness results, we find it convenient to establish existence of an impatient point in a particular subset of $X = \mathbb{I}^{\mathbb{N}}$. We introduce some auxiliary notation to present this result.

Let us denote $(0, 1)$ by Z and fix some enumeration of the rationals in Z as

$$Q = \{q_1, q_2, q_3, \cdots .\}. \tag{6}$$

For any real $r \in (0, 1)$ define the sequence $\langle a_n(r)\rangle$ by

$$a_n(r) = \begin{cases} n \text{ if } q_n \in (0, r) \\ -n \text{ if } q_n \in [r, 1). \end{cases} \tag{7}$$

and denote the set $\{a_1(r), a_2(r), ...\}$ by $\mathbb{I}(r)$. Note that $\mathbb{I}(r)$ contains infinitely many positive integers and infinitely many negative integers. We can decompose $\mathbb{I}(r)$ into pairwise disjoint sets $\mathbb{I}^+(r) = \{n \in \mathbb{I}(r) : n > 0\}$ and $\mathbb{I}^-(r) = \{n \in \mathbb{I}(r) : n < 0\}$. Moreover, since $\mathbb{I}^+(r)$ is a subset of positive integers, we can define a unique sequence of integers $\langle m_s(r)\rangle$ such that $\mathbb{I}^+(r) = \{m_1(r), m_2(r), ...\}$ and $m_1(r) < m_2(r) < \cdots$[11]. Similarly, since $\mathbb{I}^-(r)$ is a subset of negative integers (implying $\mathbb{I}^-(r)$ and every subset of $\mathbb{I}^-(r)$ has a maximum element), we can define a unique sequence of integers $\langle p_s(r)\rangle$ such that $\mathbb{I}^-(r) = \{p_1(s), p_2(s), ...\}$ and $p_1(s) > p_2(s) > \cdots$.

Proposition 2 *Suppose $Y = \mathbb{I}$ and $X = Y^{\mathbb{N}}$. Let $\succsim$ be a representable, weakly Paretian preference order on X. Then the set of points of X at which $\succsim$ exhibits impatience*

$$I = \{\mathbf{x} \in X : \ \succsim \textit{ exhibits impatience at } \mathbf{x}\}$$

[11]Set $m_1(r) = \min\{n : n \in \mathbb{I}^+(r)\}$ and define recursively for $s > 1$, $m_s(r) = \min\{n : n \in \mathbb{I}^+(r)/\{m_1(r), ..., m_{s-1}(r)\}\}$. Note that at every stage we are taking the minimum over a set of positive integers which exists.

is non-empty.

Proof. Denote by $W : X \to \mathbb{R}$ the function that represents $\succsim$ and $\mathbb{I}(r)^{\mathbb{N}}$ by $X(r)$. We will prove a stronger result. Suppose by way of contradiction, that the weakly Paretian $\succsim$ (represented by W) exhibits no impatience in $X(r)$.

Recall that Q is an fixed enumeration of the rationals in Z given by (6). For any real number $t \in Z$, there are infinitely many rational numbers from Q in $(0,t)$ and in $[t,1)$. For each real number $t \in Z$, we can then define the set $L(t) = \{n \in \mathbb{N} : q_n \in (0,t)\}$ and the sequence $\langle n_s(t)\rangle$ such that $n_1(t) < n_2(t) < n_3(t) < \cdots$ and $L(t) = \{n_1(t), n_2(t), \cdots\}$ [as in footnote 11].

Similarly, for each real number $t \in Z$, we can define the set $U(t) = \{n \in \mathbb{N} : q_n \in [t,1)\}$ and the sequence $\langle v_s(t)\rangle$ such that $v_1(t) < v_2(t) < v_3(t) \cdots$ and $U(t) = \{v_1(t), v_2(t), \cdots\}$.

For each real number $t \in (0,1)$, we note that $L(t) \cap U(t) = \emptyset$, and $L(t) \cup U(t) = \mathbb{N}$. Then, for each $r \in Z$ define the sequence $\langle x_n^r(t)\rangle_{n=1}^{\infty} = \mathbf{x}^r(t)$ as follows:

$$x_n^r(t) = \begin{cases} m_{2s-1}(r) & \text{if } n = n_s(t) \text{ for some } s \in \mathbb{N} \\ p_{2s'+1}(r) & \text{if } n = v_{s'}(t) \text{ for some } s' \in \mathbb{N}. \end{cases} \tag{8}$$

Fix $r \in Z$ for the rest of the analysis. For any $t \in Z$, and $n \in \mathbb{N}$ we must have $x_n^r(t) \in \mathbb{I}(r)$. So $\mathbf{x}^r(t) \in X(r)$ for any $t \in Z$. By way of contradiction we have assumed that W does not exhibit impatience at $\mathbf{x}^r(t)$ for any t. The proof of Proposition 4 in Dubey and Mitra (2010) shows that there is an impatient point[12] in the set

$$\hat{X}(r) = \{\mathbf{x} \in X(r) : \mathbf{x} = \mathbf{x}^r(t) \text{ as in (8) for some } t \in Z\}. \tag{9}$$

In conclusion, there is some $\mathbf{x} \in \hat{X}(r) \subset X$ at which the representable, weakly Paretian preference order $\succsim$ must exhibits impatience. ■

We can show that when period utilities belong to an order type μ subset of $\mathbb{R}$, the conclusion of Proposition 2 continues to hold.

[12]For a fixed $r \in Z$ and any $t \in Z$, note that in the sequence $\mathbf{x}^r(t)$ defined in (8) positive terms appear along some subsequence of $\mathbb{N}$ in increasing order of magnitude, and along the complementary sequence negative terms appear in decreasing order of magnitude. It is this feature of $\mathbf{x}^r(t)$ that is crucial in generalizing the proof of Proposition 4 in Dubey and Mitra (2010). The equality (26) in the proof of Proposition 4 in Dubey and Mitra (2010), holds with an inequality here. However, this modification leaves the final contradiction unaffected. For the sake of brevity, we omit the details here.

Proposition 3 *Suppose Y is a non-empty subset of $\mathbb{R}$ and is of order type μ and let $X = Y^{\mathbb{N}}$. Let $\succsim$ be a representable, weakly Paretian preference order on X. Then the set of points of X at which $\succsim$ exhibits impatience*

$$I = \{\mathbf{x} \in X : \ \succsim \ \textit{exhibits impatience at } \mathbf{x}\}$$

is non-empty.

Proof. Denote by $W : X \to \mathbb{R}$ the function that represents $\succsim$. Since Y is of order type μ, it contains a non-empty ordered subset $Y'(<)$ which is similar to $\mathbb{I}(<)$. This implies that there is a one-to-one and onto function $f : \mathbb{I} \to Y'$ that is order-preserving in the sense of condition (OP). Let $J = \mathbb{I}^{\mathbb{N}}$ and define $V : J \to \mathbb{R}$ by

$$V(z_1, z_2, ...) = W(f(z_1), f(z_2), ...) \tag{10}$$

It is easy to show that V satisfies weak Pareto. Proposition 2 implies that there is some $\mathbf{z} \in J$ at which V exhibits impatience. We will show that W exhibits impatience at $(f(z_1), f(z_2), ...)$. With some abuse of notation we will denote the sequence $(f(z_1), f(z_2), ...)$ in X by $\mathbf{f}(\mathbf{z})$. Since V exhibits impatience at $\mathbf{z} \in J$, w.l.o.g, there is some $M, N \in \mathbb{N}$ with $M > N$ such that $z_M > z_N$ and $V(\mathbf{z}(M, N)) > V(\mathbf{z})$. This information on $\mathbf{z}$ directly translates to $\mathbf{f}(\mathbf{z})$ as the function f is an order preserving map from $\mathbb{I}$ onto Y'. Hence, $f(z_M) > f(z_N)$ and $W(\mathbf{f}(\mathbf{z})(M, N)) = V(\mathbf{z}(M, N)) > V(\mathbf{z}) = W(\mathbf{f}(\mathbf{z}))$, showing that W exhibits impatience at $\mathbf{f}(\mathbf{z}) \in X$. ■

Proposition 4 (Proposition 1 in Dubey and Mitra (2010)) *Suppose Y is a non-empty subset of $\mathbb{R}$ and is **not** of order type μ and let $X = Y^{\mathbb{N}}$. Then the representable, preference order $\succsim$ defined by*

$$\mathbf{x} \succsim \mathbf{y} \textit{ iff } W(\mathbf{x}) \geq W(\mathbf{y})$$

where $W : X \to \mathbb{R}$ is given by

$$W(\mathbf{x}) = \alpha \inf\{x_n\}_{n \in \mathbb{N}} + (1 - \alpha) \sup\{x_n\}_{n \in \mathbb{N}}$$

with $\alpha \in (0, 1)$ as a parameter, does not exhibit impatience at any $\mathbf{x} \in X$ and satisfies weak Pareto.

The results in Proposition 3 and 4 imply the characterization result in Theorem 1.

7 Appendix: Robustness

This section is dedicated to proving the uncountability and genericity results of section 5. For two sets A, B if there is an injective map with domain A and range B, then we will write $A \leq_c B$. If there is a bijection (a one-to-one and onto map) with domain A and range B, then we say the sets are of the same cardinality and denote it by $A =_c B$. The Cantor-Bernstein Theorem states that $A \leq_c B$ and $B \leq_c A$ implies $A =_c B$.

Proposition 5 *Suppose $Y = \mathbb{I}$ and $X = Y^{\mathbb{N}}$. Let $\succsim$ be a representable, weakly Paretian preference order on X. Then the set of points of X at which $\succsim$ exhibits impatience*

$$I = \{\mathbf{x} \in X : \ \succsim \textit{ exhibits impatience at } \mathbf{x}\}$$

is uncountable.

Proof. Let $I(\hat{X}(r))$ be the set of impatient points of $\succsim$ in the set $\hat{X}(r)$, where $\hat{X}(r)$ is given by (9). Clearly from Proposition 2, for each $r \in Z$ the set $I(\hat{X}(r))$ is non-empty. We will first show that for $r, r' \in Z$ and $r \neq r'$, we must have $I(\hat{X}(r)) \cap I(\hat{X}(r')) = \emptyset$.

Suppose $\mathbf{x}(r) = \mathbf{x}^r(\alpha) \in I(\hat{X}(r))$ and $\mathbf{x}(r') = \mathbf{x}^{r'}(\beta) \in I(\hat{X}(r'))$ for some $\alpha, \beta \in Z$. There are two possible cases: (A) $\alpha \neq \beta$ and (B) $\alpha = \beta$. We need to show that in both cases $\mathbf{x}(r) \neq \mathbf{x}(r')$.

In case (A), assume w.l.o.g $\beta > \alpha$. Since there are infinitely many rationals in the interval (α, β) from (8) it follows that there are infinitely many natural numbers n for which

$$x_n^{r'}(\beta) > 0 > x_n^r(\alpha).$$

In particular, we must have $\mathbf{x}(r) \neq \mathbf{x}(r')$.

In case (B), assume w.l.o.g $r < r'$. Let

$$N = \min\{n \in \mathbb{N} : q_n \in [r, r')\}.$$

There are two possibilities: (i) $N = 1$ and (ii) $N > 1$. Consider $N = 1$. In this case, $m_1(r') = 1 < m_1(r)$. Since by assumption $r < r'$, if $q_n < r$ for some n, then $q_n < r'$. Hence the dominance of $m_1(r')$ over $m_1(r)$ carries over to every term, that is $m_t(r') < m_t(r)$ for all t. Since some $n = n_s(\alpha)$ exists, we must have $x_n^{r'}(\alpha) = m_{2s-1}(r') \neq m_{2s-1}(r) = x_n^r(\alpha)$. So $\mathbf{x}(r) \neq \mathbf{x}(r')$ is established when $N = 1$.

Suppose $N > 1$. Then for $i = 1, ..., N-1$ it must be that either $q_i \in (0, r)$ or $q_i \in [r', 1)$. Let there be exactly $J \geq 0$ non-negative integers $q_{J_i} \in (0, r)$ for $i = 1, ..., J$ and $K \geq 0$ non-negative integers for which $q_{K_i} \in [r', 1)$ for $i = 1, ..., K$. Of course, $J + K = N - 1$. This immediately implies that $m_j(r') = m_j(r)$ for $j = 1, ..., J$ and $p_k(r') = p_k(r)$ for $k = 1, ..., K$. However, $m_{J+1}(r') = N < m_{J+1}(r)$, and the inequality registers for all $m_j(r')$ and $m_j(r)$ for $j > J$. So, $m_j(r') < m_j(r)$ for all $j \geq J + 1$. This implies that $\mathbf{x}(r) \neq \mathbf{x}(r')$ when $N > 1$.

Since $I(\hat{X}(r)) \cap I(\hat{X}(r')) = \emptyset$, we can use the axiom of choice to define a choice function $g : Z \to X$ such that

$$g(r) \in I(\hat{X}(r)) \text{ for each } r \in Z. \tag{11}$$

So g is an injective function from g onto $g(Z) \subset I$. This shows that $Z \leq_c I$. It is well known that $Z =_c \mathbb{R}$ and $\mathbb{R} =_c \mathbb{R}^{\mathbb{N}}$. These last two equivalences are well known (see Kolmogorov and Fomin 1970, p. 15 and p. 20). These equivalences imply, $\mathbb{R}^{\mathbb{N}} \leq_c I$. Also, $\mathbb{R}^{\mathbb{N}}$ has a subset I which (trivially) has the same cardinality as I, Hence, $I \leq_c \mathbb{R}^{\mathbb{N}}$. Thus by the Cantor-Bernstein theorem, $I =_c \mathbb{R}^{\mathbb{N}}$, that is I is of the power of the continuum. ■

Proof of Theorem 2. Denote by $W : X \to \mathbb{R}$ the function that represents $\succsim$. Let $J = \mathbb{I}^{\mathbb{N}}$ and define $V : J \to \mathbb{R}$ by (10), and note that V satisfies weak Pareto. Using Proposition 2, we can claim that for each $r \in Z = (0, 1)$ there is some $\mathbf{x} \in X$ such that $\succsim$ exhibits impatience at $\mathbf{f}(\mathbf{x})$ (recall, for any $\mathbf{x} \in \mathbb{I}^{\mathbb{N}}$, $\mathbf{f}(\mathbf{x}) =(f(x_1), f(x_2), ...) \in X$), that is $\mathbf{f}(\mathbf{x}) \in I$. Denote the set of impatient points of V by I_V. For any $\mathbf{x} \in I_V$, we have $\mathbf{f}(\mathbf{x}) \in I$ and by (OP) the function $F : I_V \to I$ defined by $F(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ is one-to-one and onto on its image set, $F(I_V)$, and $F(I_V) \subset I$. So I has a subset, $F(I_V)$, that has the same cardinality as I_V, so $I_V \leq_c I$. Since by Proposition 5, $I_V =_c \mathbb{R}^{\mathbb{N}}$ we must have $\mathbb{R}^{\mathbb{N}} \leq_c I$. Also, $\mathbb{R}^{\mathbb{N}}$ has a subset I which (trivially) has the same cardinality as I, so $I \leq_c \mathbb{R}^{\mathbb{N}}$. By the Cantor-Bernstein theorem, $I =_c \mathbb{R}^{\mathbb{N}}$, that is I is of the power of the continuum. ■

Proof of Thorem 3. We have to show that for any $\mathbf{x} \in X$ and $\epsilon > 0$ the open ball of radius ϵ and center $\mathbf{x}$ (denoted $B(\mathbf{x}, \epsilon)$) has non-empty intersection with I. Let $\bar{\mathbf{x}} \in X$ and $\epsilon > 0$ we need to show $B(\bar{\mathbf{x}}, \epsilon) \cap I \neq \emptyset$. Let $a \equiv \liminf_{n \to \infty}\{\bar{x}_n\}$.

Given a fixed $\epsilon > 0$ assume w.l.o.g, $(a-\epsilon/2, a+\epsilon/2) \subset Y$ and denote the interval $(a-\epsilon/2, a+\epsilon/2)$ by Y'. From the definition of a it follows that there is some N such that for all $k > N$ we must have

$$a - (\epsilon/2) < \bar{x}_k. \tag{12}$$

Choose N to be the smallest such natural number for which (12) holds and fix it. We will now recursively define a particular subsequence of the natural numbers $\{N_1, N_2, ...\}$ such that $\bar{x}_{N_k} < a + (\epsilon/2)$ for all $k \in \mathbb{N}$. As step 1, set $N_1 = N$ and find the smallest natural number $N_2 > N$ such that $\bar{x}_{N_2} < a+(\epsilon/2)$. In step 2, start with N_2 and find the smallest natural number $N_3 > N_2$ such that $\bar{x}_{N_3} < a + (\epsilon/2)$. Proceed recursively to obtain the sequence $\{N_1, N_2, ...\}$ such that

$$\bar{x}_{N_k} < a + (\epsilon/2) \text{ for all } k \in \mathbb{N}. \tag{13}$$

Claim 3. *If $b \in Y'$, then $\bar{x}_{N_k} - \epsilon < b < \bar{x}_{N_k} + \epsilon$ for all $k \in \mathbb{N}$*: Observe that by subtracting ϵ from both sides of (13) and comparing with b, we get $\bar{x}_{N_k} - \epsilon < a - (\epsilon/2) < b$ for all $k \in \mathbb{N}$. Furthermore, by adding ϵ to both sides of (12) and comparing with b we get $b < a+(\epsilon/2) < \bar{x}_k+\epsilon$ for all $k \in \mathbb{N}$ as was needed. □

Denote X' by $(Y')^{\mathbb{N}}$ and let $g : X' \to X$ be defined as $g(\mathbf{y}) = (g_k(\mathbf{y}))_{k\in\mathbb{N}}$ where,

$$g_k(\mathbf{y}) = \begin{cases} y_i & \text{if } k = N_i \text{ for some } i \\ \bar{x}_k & \text{otherwise.} \end{cases} \tag{14}$$

Define the function $U : X' \to \mathbb{R}$ as $U(\mathbf{y}) = W(g(\mathbf{y}))$ for $\mathbf{y} \in X'$.

We show first that U satisfies weak Pareto on X'. To verify weak Pareto, take $\mathbf{y}' >> \mathbf{y}$ for $\mathbf{y}, \mathbf{y}' \in X'$ and note that (14) implies $g_k(\mathbf{y}') > g_k(\mathbf{y})$ for $k = N_i$ for $i \in \mathbb{N}$. Since W satisfies infinite Pareto, the following must be true

$$U(\mathbf{y}') = W(g(\mathbf{y}')) > W(g(\mathbf{y})) = U(\mathbf{y}),$$

establishing that U satisfies weak Pareto on X'. As Y' is a set of order type μ, Theorem 1 implies that there exists some $\mathbf{y}' \in X'$ at which U exhibits impatience. So, there is some $m, n \in \mathbb{N}$ with $m > n$ such that

$$(y'_n - y'_m)(U(\mathbf{y}'(m, n)) - U(\mathbf{y}')) < 0. \tag{15}$$

The information in (15) can be translated to obtain an impatient point in $B(\bar{x}, \epsilon)$. From the definition of g we must have $N_n < N_m$, $g_{N_n}(\mathbf{y}') = y'_n$ and $g_{N_m}(\mathbf{y}') = y'_m$. Using (15) we get

$$(g_{N_n}(\mathbf{y}') - g_{N_m}(\mathbf{y}'))[W(g(\mathbf{y}')(N_m, N_n)) - W(g(\mathbf{y}'))] < 0.$$

Hence, W must exhibit impatience at $g(\mathbf{y}') \in B(\bar{x}, \epsilon)$ (by Claim 3, and (14) we must have $g(\mathbf{y}) \in B(\bar{x}, \epsilon)$ for all $\mathbf{y} \in X'$) as was required. ■

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